Locally, $J^1(M,N)$ looks like $U \times V \times M_{m,n}(\mathbb{R})$ and $S^r := U \times V \times M^r_{m,n}(\mathbb{R})$ is a submf. Given f smooth we have

$$\Sigma^i(f) = (j^1 f)^{-1}(S^{m-i}) \quad \text{if } m \geq n$$
$$\text{or } (S^{n-i}) \quad \text{if } m \leq n \quad !$$

<u>Recap:</u> Jets & Jet spaces / bundles

- $f \sim_0 g$ at $x \iff f(x) = g(x)$
- $f \sim_1 g$ at $x \iff f \sim_0 g$ at x and $df_x = dg_x$
- $f \sim_2 g$ at $x \iff f \sim_0 g$ at x and $df \sim_1 dg$ at $(x,v) \in T_x M$

  .... $f \sim_k g$ at x : Equivalence relation

- $J^k(M,N)_{x,y} := \{ f \in C^\infty(M,N) \mid f(x) = y \} / \sim_k$ at x

- $J^k(M,N) := \bigcup_{x,y \in M \times N} J^k(M,N)_{x,y}$ $\quad$ $\boxed{\text{Q: What's the fiber?}}$

$$\begin{tikzcd} & J^k(M,N) \arrow[dl, "s"'] \arrow[d, "t"] \arrow[dr, "s\times t"] \arrow[drr, "\pi^k_\ell"] & & \\ M & N & M\times N & J^\ell(M,N)_{x,y} \end{tikzcd} \quad (\text{for } \ell < k)$$

- k-jet of $f: M \to N$: $j^k f : M \to J^k(M,N)$
$$x \mapsto [f] \in J^k(M,N)_{x, f(x)}$$

locally, $j^k f(x) = \Big(x_1,\dots,x_m,\ \underbrace{y_1,\dots,y_n}_{=f(x)},\ \Big\{\frac{\partial f_i}{\partial x_j}\Big\}_{\substack{i=1,\dots,n\\ j=1,\dots,m}},\ \Big\{\frac{\partial^2 f_i}{\partial x_j \partial x_l}\Big\}_{\substack{i=1,\dots,n\\ 1\le j\le l\le m}},\ \dots \Big)$

<u>Thm:</u> $J^k(M,N)$ is smooth <u>fin. dim.</u> manifold

$j^k f : M \to J^k(M,N)$ is smooth

$m + n\left(\frac{(m+k)!}{m!\,k!}\right)$

$\rightsquigarrow$ generalizations:

- $J^\infty(M,N)$
- algebraic geometry: $J^k(M)$ allows to study singular <u>points</u> of M
- Jets of sections of vector bundles: Study differential operators $\Gamma\left(\begin{smallmatrix}E_1\\ \downarrow\\ M\end{smallmatrix}\right) \to \Gamma\left(\begin{smallmatrix}E_2\\ \downarrow\\ M\end{smallmatrix}\right)$

We need one last ingredient to state the (strong) transversality theorem.

IV. Whitney topologies

For $U \subset J^k(M,N)$ let

$$M(U) := \{ f \in C^\infty(M,N) \mid j^k f(M) \subset U \}.$$

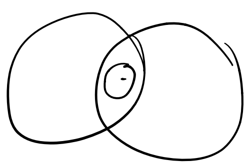
1. Def

- The Whitney C^k-topology is the topology on $C^\infty(M,N)$ generated by the basis $\{M(U)\}_{U \subset J^k(M,N) \text{ open}}$.

Write $\mathcal{W}_k$ for the set of open sets in this topology.

⌈Recall: Basis of topology on X is $\mathcal{B} \subset P(X)$

s.t. 1. $\bigcup_{B \in \mathcal{B}} B = X$



2. $B_1, B_2 \in \mathcal{B}$: $\forall x \in B_1 \cap B_2$: $\exists B_3 \in \mathcal{B}$ with $x \in B_3 \wedge B_3 \subset B_1 \cap B_2$

This generates a topology by

open sets := unions of elements of $\mathcal{B}$

eg Intervalls on $\mathbb{R}$

$\lrcorner$

- The Whitney C^∞-topology on $C^\infty(M,N)$ is generated by

$$\mathcal{W} := \bigcup_{k=0}^{\infty} \mathcal{W}_k .$$

(this is well-defined because

$$\ell \leq k \Rightarrow \mathcal{W}_\ell \subset \mathcal{W}_k$$
)

How do open nbhoods look like in the Whitney topologies?

Let $f \in C^\infty(M,N)$ and let d a metric on $J^k(M,N)$ inducing its topology ("every mf is metrizable") and $\delta : M \to \mathbb{R}_+$ continuous.

Then

$$B_\delta(f) := \{ g \in C^\infty(M,N) \mid \forall x \in M : d(j^k f(x), j^k g(x)) < \delta(x) \}$$

is open:

$$\Delta : J^k(M,N) \to \mathbb{R}, \quad \sigma \mapsto \delta(s(\sigma)) - d(j^k f(s(\sigma)), \sigma)$$

is continuous. Set $U := \Delta^{-1}(\mathbb{R}_+)$.

This is open and $B_\delta(f) = \underbrace{M(U)}_{= \{g \mid j^k g(M) \subset U\}}$

If M is compact, then $B_{\frac{1}{n}}(f)$ defines a countable neighborhood basis (each δ is bounded below by some nr $\frac{1}{n}$ ...)

and $\quad f_n \to f$ in Whitney C^k-topology on $C^\infty(M,N)$

$\iff \quad j^k f_n \to j^k f$ uniformly in $C^\infty(M, J^k(M,N))$

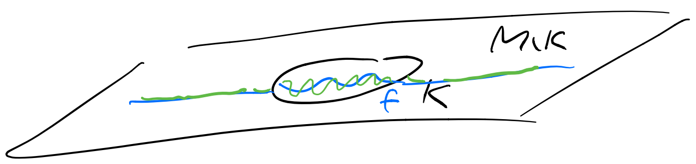
If M is not compact, then there is no countable nbh basis and

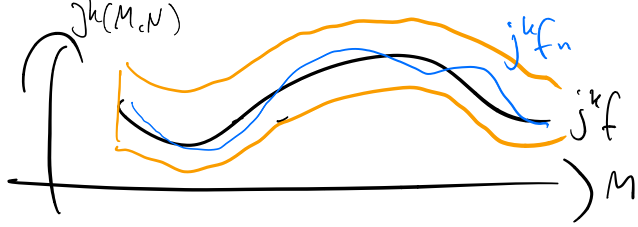
$f_n \to f$ in Whitney C^k-topology on $C^\infty(M,N)$

$\iff \quad \exists\, K \subset M$ compact such that

$(\ast)$ on K: $\quad j^k f_n \to j^k f$ uniformly

off K: $\quad \exists\, n_0 \in \mathbb{N} : \forall n \geq n_0 : f_n(x) = f(x) \quad \forall x \in M \setminus K$





Proof: $\Leftarrow$ ✓ $\Rightarrow$:

Let $f_n \to f$ in Whitney C^k-topology and assume that there is no K s.t. $(*)$.

Take a sequence of compacta $\{K_i\}_{i\in\mathbb{N}}$ s.t.

$K_i \subset \mathring{K}_{i+1}$ and $M = \bigcup_{i\in\mathbb{N}} K_i$

$\exists\, n_1$ with $f_{n_1} \neq f \Rightarrow \exists\, x_1 \in M : d(j^k f_{n_1}(x_1), j^k f(x_1)) =: a_1 > 0$

$\exists\, m_1$ with $x \in K_{m_1}$.

Set $\delta|_{K_{m_1}} \equiv a_1$.

Repeat to get after s steps:

$n_1 < \dots < n_s$, K_{m_s}, $\delta : K_{m_s} \to \mathbb{R}_+$

and $x_1, \dots, x_s \in K_{m_s}$ with

$$\forall i \leq s : \quad d(j^k f_{n_i}(x_i), j^k f(x_i)) > \delta(x_i).$$

Now choose $f_{n_{s+1}}$ for $n_{s+1} > n_s$

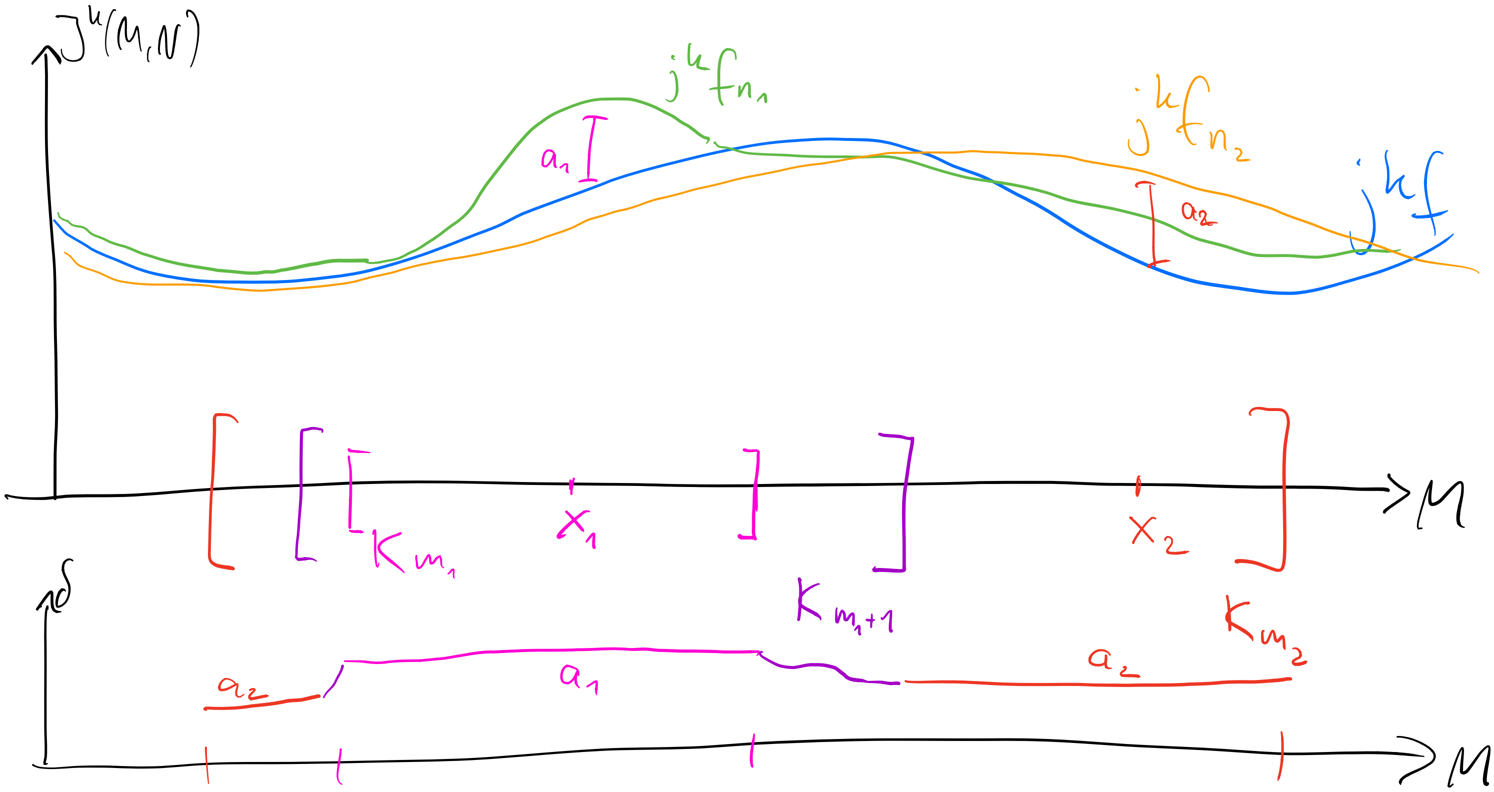
such that $f_{n_{s+1}} \neq f$ off K_{m_s+1}.

Let x_{s+1} lie outside K_{m_s+1} with

$$d\left(j^k f_{n_{s+1}}(x_{s+1}), \, j^k f(x_{s+1}) \right) =: a_{s+1} > 0 .$$

Choose m_{s+1} such that $x_{s+1} \in K_{m_{s+1}}$ and extend δ (continuously) to $K_{m_{s+1}}$ by

$$\delta|_{K_{m_{s+1}} \setminus K_{m_s+1}} = a_{s+1} .$$



This procedure gives a subsequence f_{n_i} and $\delta : M \to \mathbb{R}_+$ contin. such that

$$\forall i : \quad f_{n_i} \notin B_\delta(f)$$ which contradicts $f_n \to f$ ↯

Why are these topologies useful?

Because

2. Def X top. space.

- $Y \subset X$ is **residual** if it is a countable intersection of open and dense subsets of X.

- X is a **Baire space** if every residual set is dense.

eg

- $\mathbb{Q} \subset \mathbb{R}$ is not residual
- $\mathbb{R} \setminus \mathbb{Q} \subset \mathbb{R}$ is residual

  $q : \mathbb{N} \to \mathbb{Q}$ bij. then $\mathbb{R} \setminus \mathbb{Q} = \bigcap_{n \in \mathbb{N}} (\mathbb{R} \setminus \{q_n\})$

- Cantor set C is not a residual subset of $\mathbb{R}$, but $\mathbb{R} \setminus C$ is.

- $\mathbb{R}$ is a Baire space
- C is a Baire space

3. <u>Def</u> X (Baire) space. $P: X \to \{0,1\}$ a "property". P is generic if $P^{-1}(1)$ contains a residual set in X.

<u>eg</u> Morse functions: $X = C^\infty(M,\mathbb{R})$, $P(f) =$ f is Morse. Then P is generic (of course, this needs to be shown)

4. <u>Prop</u>:

Let M, N smooth mfs. Then $C^\infty(M,N)$ is a Baire space in the Whitney C^∞ topology.

<u>Proof</u>: Lit. (G&G)

<u>eg</u> $\{\text{Morse functions}\} \subset C^\infty(M,\mathbb{R})$ residual & $C^\infty(M,\mathbb{R})$ is Baire, thus $\{\text{Morse fctns}\}$ is a dense subset.